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Discrepancy and Integration of Continuous Functions

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1. INTRODUCTION

A sequence $p_1, p_2, ..., p_N$ of nonnegative numbers is said to be a weight sequence if

$$\sum_{\kappa=1}^{N} p_\kappa = 1.$$

Let $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ be a sequence of N points in the s-dimensional unit cube $E^s = [0, 1]^s$ and $p_1, p_2, ..., p_N$ be a weight sequence. We call the numbers $p_1, p_2, ..., p_N$ weights of the points $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$, respectively. For a subset A of E^s, denote by $\chi(A; \bar{x})$ and $\mu(A)$ the characteristic function and Lebesgue measure of A, respectively, and put

$$T_N(A) = \sum_{\kappa=1}^{N} p_\kappa \chi(A; \bar{a}_\kappa).$$

For a point $\bar{x} = (x_1, ..., x_s)$ lying in E^s, write

$$G_{\bar{x}} = [0, x_1) \times \cdots \times [0, x_s).$$

We recall that the number

$$D_N = \sup_{\bar{x} \in E^s} |\mu(G_{\bar{x}}) - T_N(G_{\bar{x}})| \tag{1}$$

is called the discrepancy of the sequence $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ with respect to the weights $p_1, p_2, ..., p_N$.

We consider in this paper quadrature formulae of the type

$$\int_{E^s} f(\bar{x})\, d\bar{x} = \sum_{\kappa=1}^{N} p_\kappa f(\bar{a}_\kappa) + R_N(f) \tag{2}$$

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for integration of continuous functions on E^s. Denote by $C(A)$ the set of all continuous functions, defined on a subset A of E^s. As a characteristic of a function $f \in C(E^s)$ we use its modulus of continuity

$$\omega(f; \delta) = \sup_{\substack{\|\bar{x} - \bar{y}\| \leqslant \delta \\ \bar{x}, \bar{y} \in E^s}} |f(\bar{x}) - f(\bar{y})|, \qquad \delta \geqslant 0,$$

where $\|\bar{x}\|$ denotes the maximum norm of a point $\bar{x} = (x_1, ..., x_s)$ lying in $\mathbb{R}^s$, i.e.,

$$\|\bar{x}\| = \max_{1 \leqslant j \leqslant s} |u_j|.$$

We recall that the inequality

$$\omega(f; \delta_1 + \delta_2) \leqslant \omega(f; \delta_1) + \omega(f; \delta_2) \tag{3}$$

holds for all $\delta_1, \delta_2 \geqslant 0$.

In the one-dimensional case, it is known that if $f \in C(E)$ then

$$|R_N(f)| \leqslant \omega(f; D_N). \tag{4}$$

This estimate was proved by Niederreiter [1] for the weights

$$p_1 = p_2 = \cdots = p_N = 1/N. \tag{5}$$

The estimate (4) for arbitrary weights was obtained by the author [2] and reobtained by Niederreiter and Tichy [3].

Now let $s \geqslant 2$ and $f \in C(E^s)$. In the case (5), Hlawka [4] proved the estimate

$$|R_N(f)| \leqslant (2^{2s-1} + 1)\, \omega(f; [D_N^{-1}]^{-1/s}).$$

Here and throughout, $[\alpha]$ denotes the integral part of a real number α. Shi Shu-zhong [5, 6] (also for the case of equal weights), answering to a question of Niederreiter, proved the estimate

$$|R_N(f)| \leqslant 5\omega(f; D_N^{1/s}).$$

For arbitrary weights Totkov [7] established the estimate

$$|R_N(f)| \leqslant (2^s + 1)\, \omega(f; 2[D_N^{-1}]^{-1/s}).$$

The purpose of the present paper is to prove the following two results.

THEOREM 1. *Suppose $s \geqslant 2$. Let D_N be the discrepancy of a sequence $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ in E^s with respect to the weights $p_1, p_2, ..., p_N$. Then for every $f \in C(E^s)$, we have*

$$|R_N(f)| \leqslant 4\omega(f; D_N^{1/s}). \tag{6}$$

THEOREM 2. *Suppose $s \geqslant 1$. Let D_N be the discrepancy of a sequence $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ in E^s with respect to the weights $p_1, p_2, ..., p_N$. Suppose that c is a positive number and the estimate*

$$|R_N(f)| \leqslant c\omega(f; D_N^{1/s}) \tag{7}$$

holds for every $f \in C(E^s)$. Then

$$c \geqslant 1. \tag{8}$$

Now, denote by c_0 the minimal possible constant c for which the estimate (7) holds for every $f \in C(E^s)$. It follows from the above theorems that

$$1 \leqslant c_0 \leqslant 4.$$

The exact value of c_0 is unknown. We think that

$$c_0 = 1$$

but we cannot prove that. So we set the following

PROBLEM. Check the validity of the following assertion: Suppose $s \geqslant 2$ and D_N is the discrepancy of a sequence $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ in E^s with respect to the weights $p_1, p_2, ..., p_N$. Then for every $f \in C(E^s)$, we have

$$|R_N(f)| \leqslant \omega(f; D_N^{1/s}). \tag{9}$$

Note that in the one-dimensional case ($s = 1$), the estimate (9) coincides with (4) and so it is true.

2. AUXILIARY RESULTS

To prove Theorem 1 we need some lemmas. In what follows, we use the following notations. For $f \in C(E^s)$, we write

$$\|f\|_C = \sup_{\bar{x} \in E^s} |f(\bar{x})|.$$

Let $Q=[0,1)\times[\alpha_2,\beta_2)\times\cdots\times[\alpha_s,\beta_s)$ be a subinterval of E^s. Define the quantity $d(Q)$ by

$$d(Q)=\max_{2\leqslant j\leqslant s}|\beta_j-\alpha_j|.$$

LEMMA 1. *Let $a_1, a_2, ..., a_N$ be a nondecreasing sequence in $E=[0,1]$. Then its discrepancy D_N with respect to arbitrary weights $p_1, p_2, ..., p_N$ is given by*

$$D_N=\max_{1\leqslant\kappa\leqslant N}\max\{|a_\kappa-b_\kappa|,|a_\kappa-b_{\kappa-1}|\},$$

where the numbers $b_0, b_1, ..., b_N$ are defined by

$$b_0=0,\qquad b_\kappa=\sum_{i=1}^{\kappa}p_i\quad(\kappa=1,2,...,N). \tag{10}$$

In the special case (5), this lemma was proved by Niederreiter [8]. In the general case, it was obtained by the author [2] and reobtained by Niederreiter and Tichy [3].

The next assertion is a consequence of Lemma 1.

LEMMA 2. *Let $a_1, a_2, ..., a_N$ be a nondecreasing sequence in E and D_N be its discrepancy with respect to the weights $p_1, p_2, ..., p_N$. Define the numbers $b_0, b_1, ..., b_N$ by* (10). *Then for every $x\in[b_{\kappa-1}, b_\kappa]$, $1\leqslant\kappa\leqslant N$, we have*

$$|x-a_\kappa|\leqslant D_N.$$

LEMMA 3. *Let $Q_\gamma=[0,\gamma)\times[\alpha_2,\beta_2)\times\cdots\times[\alpha_s,\beta_s)$ be a subinterval of E^s. Suppose that $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ is a sequence in E^s and D_N is its discrepancy with respect to the weights $p_1, p_2, ..., p_N$. Then*

$$|\mu(Q_\gamma)-T_N(Q_\gamma)|\leqslant 2^{s-1}D_N.$$

Proof. The statement of the lemma follows from the definition of D_N and the inequality

$$|\mu(Q_\gamma)-T_N(Q_\gamma)|\leqslant\sum_{\bar{x}}|\mu(G_{\bar{x}})-T_N(G_{\bar{x}})|,$$

where the sum is over all points $\bar{x}=(x_1, ..., x_s)$ with $x_1=\gamma$ and $x_j=\alpha_j$ or β_j for $j=2, ..., s$.

LEMMA 4. *Let $Q=[0,1)\times[\alpha_2,\beta_2)\times\cdots\times[\alpha_s,\beta_s)$ and $Q_\gamma=[0,\gamma)\times[\alpha_2,\beta_2)\times\cdots\times[\alpha_s,\beta_s)$ be subintervals of E^s. Suppose that $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ is a* *sequence in E^s and D_N is its discrepancy with respect to the weights $p_1, p_2, \ldots, p_N$. Then*

$$|\gamma T_N(Q) - T_N(Q_\gamma)| \leqslant 2^s D_N.$$

Proof. Note that $\mu(Q_\gamma) = \gamma\mu(Q)$ and $Q = Q_\gamma$ if $\gamma = 1$. Hence, we get from Lemma 3,

$$\begin{aligned} &|\gamma T_N(Q) - T_N(Q_\gamma)| \\ &\quad \leqslant |\gamma T_N(Q) - \mu(Q_\gamma)| + |\mu(Q_\gamma) - T_N(Q_\gamma)| \\ &\quad = \gamma\, |\mu(Q) - T_N(Q)| + |\mu(Q_\gamma) - T_N(Q_\gamma)| \\ &\quad \leqslant \gamma 2^{s-1} D_N + 2^{s-1} D_N \leqslant 2^s D_N. \end{aligned}$$

LEMMA 5. *Let $Q = [0, 1) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_s, \beta_s)$ be a subinterval of E^s. Suppose that $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is a sequence in E^s and $q_1, q_2, \ldots, q_n$ are arbitrary weights. Denote by D'_n the discrepancy of the one-dimensional sequence formed by the first coordinates of $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ with respect to the weights $q_1, q_2, \ldots, q_n$. Then for every $f \in C(Q)$, we have*

$$\left| \int_Q f(\bar{x})\, d\bar{x} - \mu(Q) \sum_{\kappa=1}^{n} q_\kappa f(\bar{a}_\kappa) \right| \leqslant \mu(Q)\, \omega(f; d_n(Q)), \tag{11}$$

where

$$d_n(Q) = \max\{D'_n, d(Q)\}. \tag{12}$$

Proof. Without loss of generality we may assume that the points $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ are ordered in such a way that the sequence of their first coordinates in nondecreasing. For each $\kappa = 1, 2, \ldots, n$, put

$$Q_\kappa = [b_{\kappa-1}, b_\kappa) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_s, \beta_s),$$

where

$$b_0 = 0, \qquad b_\kappa = \sum_{i=1}^{\kappa} q_i \quad (\kappa = 1, 2, \ldots, n).$$

It is obvious that the subintervals $Q_1, Q_2, \ldots, Q_n$ define a subdivision of Q and

$$\int_{Q_\kappa} d\bar{x} = \mu(Q_\kappa) = \mu(Q)\, q_\kappa \qquad (\kappa = 1, 2, \ldots, n).$$

Therefore,

$$\left|\int_Q f(\bar{x})\,d\bar{x} - \mu(Q) \sum_{\kappa=1}^{n} q_\kappa f(\bar{a}_\kappa)\right|$$

$$= \left| \sum_{\kappa=1}^{n} \int_{Q_\kappa} (f(\bar{x}) - f(\bar{a}_\kappa))\,d\bar{x} \right|$$

$$\leqslant \sum_{\kappa=1}^{n} \int_{Q_\kappa} |f(\bar{x}) - f(\bar{a}_\kappa)|\,d\bar{x}$$

$$\leqslant \sum_{\kappa=1}^{n} \int_{Q_\kappa} \omega(f; \|\bar{x} - \bar{a}_\kappa\|)\,dx. \tag{13}$$

It follows from Lemma 2 that for every $x \in Q_\kappa$ $(1 \leqslant \kappa \leqslant n)$, we have

$$\|\bar{x} - \bar{a}_\kappa\| \leqslant \max(D'_n, d(Q)) = d_n(Q). \tag{14}$$

Combining (13) and (14) we get (11).

LEMMA 6. *Let D_N be the discrepancy of a sequence $\bar{a}_1, \bar{a}_2, ..., \bar{a}_N$ in E^s with respect to the weights $p_1, p_2, ..., p_N$. Suppose that $\mathscr{A}$ is a subdivision of E^s into subintervals of the type $[0, 1) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_s, \beta_s)$ and $T_N(Q) > 0$ for every $Q \in \mathscr{A}$. Then*

$$|R_N(f)| \leqslant 2^{s-1}\,|\mathscr{A}|\,D_N\,\|f\|_C + \sum_{Q \in \mathscr{A}} \mu(Q)\,\omega(f; \delta_N(Q)),$$

where $|\mathscr{A}|$ denotes the number of elements of $\mathscr{A}$ and

$$\delta_N(Q) = \max\left\{\frac{2^s D_N}{T_N(Q)}, d(Q)\right\}. \tag{15}$$

Proof. It is easy to see that

$$|R_N(f)| \leqslant \sum_{Q \in \mathscr{A}} |R_N(Q; f)|, \tag{16}$$

where

$$R_N(Q; f) = \int_Q f(\bar{x})\,d\bar{x} - \sum_{\kappa=1}^{N} p_\kappa f(\bar{a}_\kappa)\,\chi(Q; \bar{a}_\kappa). \tag{17}$$

Let $Q = [0, 1) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_s, \beta_s)$ be an arbitrary element of $\mathscr{A}$. We shall prove at first that

$$|R_N(Q; f)| \leqslant \mu(Q)\,\omega(f; \delta_N(Q)) + 2^{s-1} D_N\,\|f\|_C. \tag{18}$$

Denote by n the number of points $\bar{a}_\kappa$ ($\kappa = 1, 2, ..., N$) lying in Q. Without loss of generality we may assume that these points are $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$. Then

$$T_N(Q) = \sum_{\kappa=1}^{n} p_\kappa$$

and the sequence

$$q_\kappa = p_\kappa / T_N(Q), \qquad \kappa = 1, 2, ..., n$$

is a weight sequence. Denote by D'_n the discrepancy of the one-dimensional sequence formed by the first coordinates of $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$ with respect to the weights $q_1, q_2, ..., q_n$. Obviously,

$$\begin{aligned} D'_n &= \sup_{0 \leqslant \gamma \leqslant 1} \left| \gamma - \sum_{\kappa=1}^{n} q_\kappa \chi(Q_\gamma; \bar{a}_\kappa) \right| \\ &= T_N(Q)^{-1} \sup_{0 \leqslant \gamma \leqslant 1} |\gamma T_N(Q) - T_N(Q_\gamma)|, \end{aligned}$$

where $Q_\gamma = [0, \gamma) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_s, \beta_s)$. Now, it follows from Lemma 4 that

$$D'_n \leqslant \frac{2^s D_N}{T_N(Q)}. \tag{19}$$

Using Lemmas 3 and 5 we get from (17),

$$\begin{aligned} |R_N(Q; f)| &\leqslant \left| \int_Q f(\bar{x})\, d\bar{x} - \mu(Q) \sum_{\kappa=1}^{n} q_\kappa f(\bar{a}_\kappa) \right| \\ &\quad + \left| \mu(Q) \sum_{\kappa=1}^{n} q_\kappa f(\bar{a}_\kappa) - \sum_{\kappa=1}^{n} p_\kappa f(\bar{a}_\kappa) \right| \\ &\leqslant \mu(Q)\, \omega(f; d_n(Q)) \\ &\quad + T_N(Q)^{-1} |\mu(Q) - T_N(Q)| \left| \sum_{\kappa=1}^{n} p_\kappa f(\bar{a}_\kappa) \right| \\ &\leqslant \mu(Q)\, \omega(f; d_n(Q)) + 2^{s-1} D_N \|f\|_C, \end{aligned} \tag{20}$$

where $d_n(Q)$ is defined by (12). From (19) and (20), we obtain (18). Now the conclusion of the lemma follows from (16) and (18).

The next assertion was established in [6]. One can easily check its validity.

LEMMA 7. *Let $f \in C(E^s)$ and the function $\tilde{f}$ is defined on E^s by*

$$\tilde{f}(\bar{x}) = f(\bar{x}) - f(\bar{a}),$$

where $a = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$. Then

(i) $R_N(\tilde{f}) = R_N(f)$,

(ii) $\omega(\tilde{f}; \delta) = \omega(f; \delta)$ *for* $\delta \geqslant 0$,

(iii) $\|\tilde{f}\|_C \leqslant \omega(\tilde{f}; \frac{1}{2})$.

3. PROOF OF THEOREM 1

Suppose at first that $D_N^{1/s} \geqslant \frac{1}{4}$. It follows from Lemma 5 that

$$|R_N(f)| \leqslant \omega(f; 1).$$

Since $1 \leqslant 4D_N^{1/s}$, we get from this

$$|R_N(f)| \leqslant \omega(f; 4D_N^{1/s}). \tag{21}$$

Now let $D_N^{1/s} \leqslant \frac{1}{4}$. Put

$$m = \left[\frac{1}{2D_N^{1/s}}\right].$$

It is easy to show that for every real number α $(\alpha \geqslant 2)$,

$$\tfrac{2}{3}\alpha \leqslant [\alpha] \leqslant \alpha.$$

Applying this inequality with $\alpha = 1/2D_N^{1/s}$ we obtain

$$2D_N^{1/s} \leqslant \frac{1}{m} \leqslant 3D_N^{1/s}. \tag{22}$$

Denote by $\mathscr{A}$ the subdivision of E^s into subintervals of the type

$$[0, 1] \times \left[\frac{\kappa_2 - 1}{m}, \frac{\kappa_2}{m}\right) \times \cdots \times \left[\frac{\kappa_s - 1}{m}, \frac{\kappa_s}{m}\right),$$

where κ_j are integers with $1 \leqslant \kappa_j \leqslant m$ $(j = 2, ..., s)$. Let Q be an arbitrary element of $\mathscr{A}$. According to Lemma 3,

$$|\mu(Q) - T_N(Q)| \leqslant 2^{s-1} D_N.$$

Hence, we get from (22)

$$T_N(Q) \geqslant \mu(Q) - 2^{s-1} D_N$$
$$= \left(\frac{1}{m}\right)^{s-1} - 2^{s-1} D_N \geqslant 2^{s-1} D_N^{1-1/s} - 2^{s-1} D_N$$
$$= 2^{s-1} D_N (D_N^{-1/s} - 1). \tag{23}$$

Since $D_N^{1/s} \leqslant \frac{1}{4}$, it follows that

$$D_N^{-1/s} - 1 \geqslant \tfrac{3}{4} D_N^{-1/s}. \tag{24}$$

Hence, we get from (23) and (24),

$$T_N(Q) \geqslant \tfrac{3}{8} 2^s D_N^{1-1/s}. \tag{25}$$

Define $\delta_N(Q)$ by (15). Then from (25) and (22), we obtain

$$\delta_N(Q) = \max\left\{\frac{2^s D_N}{T_N(Q)}, \frac{1}{m}\right\} \leqslant 3 D_N^{1/s}. \tag{26}$$

Now, taking into account that $|\mathscr{A}| = m^{s-1}$, we deduce from (22), (26), and Lemma 6 the following estimate

$$|R_N(f)| \leqslant (2m)^{s-1} D_N \|f\|_C + \sum_{Q \in \mathscr{A}} \mu(Q)\, \omega(f; 3D_N^{1/s})$$
$$= (2m)^{s-1} D_N \|f\|_C + \omega(f; 3D_N^{1/s})$$
$$\leqslant (2m)^{-1} \|f\|_C + \omega(f; 3D_N^{1/s}). \tag{27}$$

It follows from Lemma 7 that, without loss of generality, we may assume

$$\|f\|_C \leqslant \omega(f; \tfrac{1}{2}).$$

Then using (3) and (22) we deduce

$$\|f\|_C \leqslant 2m\omega\left(f; \frac{1}{4m}\right) \leqslant 2m\omega(f; \tfrac{3}{4} D_N^{1/s}). \tag{28}$$

Combining (27) and (28) we get

$$|R_N(f)| \leqslant \omega(f; \tfrac{3}{4} D_N^{1/s}) + \omega(f; 3D_N^{1/s}). \tag{29}$$

From (21), (29), and (3), it follows that in both cases

$$|R_N(f)| \leqslant \omega(f; D_N^{1/s}) + \omega(f; 3D_N^{1/s}). \tag{30}$$

Now, the estimate (6) follows from (30) and (3). This completes the proof of Theorem 1.

4. Proof of Theorem 2

Suppose that the estimate (7) hold for every $f \in C(E^s)$. Choose a positive real number ε with

$$0 < \varepsilon < \min\{\delta_N, D_N^{1/s}\},$$

where

$$\delta_N = \min_{\substack{1 \leqslant i < j \leqslant N \\ \bar{a}_i \neq \bar{a}_j}} \|\bar{a}_i - \bar{a}_j\|.$$

We denote by $U_\varepsilon(\bar{a})$ the $\varepsilon/2$-neighborhood of a point $\bar{a} \in \mathbb{R}^s$, i.e.,

$$U_\varepsilon(\bar{a}) = \{\bar{x} \in \mathbb{R}^s \colon \|\bar{x} - \bar{a}\| \leqslant \varepsilon/2\}.$$

Define the function f_ε on $\mathbb{R}^s$ by

$$f_\varepsilon(\bar{x}) = \begin{cases} \dfrac{2}{\varepsilon}\|\bar{x} - \bar{a}_\kappa\|; & \bar{x} \in U_\varepsilon(\bar{a}_\kappa),\ \kappa = 1, 2, ..., N. \\ 1 & \text{otherwise.} \end{cases}$$

Obviously,

$$f_\varepsilon \in C(E^s)$$

and

$$f(\bar{a}_\kappa) = 0 \qquad \text{for} \quad \kappa = 1, 2, ..., N.$$

Consequently,

$$\begin{aligned} R_N(f_\varepsilon) &= \int_{E^s} f_\varepsilon(\bar{x})\, d\bar{x} \\ &= 1 - \int_{E^s} (1 - f_\varepsilon(\bar{x}))\, d\bar{x} \\ &\geqslant 1 - \sum_{\kappa=1}^{N} \int_{U_\varepsilon(\bar{a}_\kappa)} (1 - f_\varepsilon(\bar{x}))\, d\bar{x} \\ &= 1 - N\varepsilon^s + \frac{2}{\varepsilon} \sum_{\kappa=1}^{N} \int_{\|\bar{x} - \bar{a}_\kappa\| \leqslant \varepsilon/2} \|\bar{x} - \bar{a}_\kappa\|\, d\bar{x} \\ &= 1 - N\varepsilon^s + \frac{2N}{\varepsilon} \int_{\|\bar{x}\| \leqslant \varepsilon/2} \|\bar{x}\|\, d\bar{x} \\ &= 1 - \frac{N\varepsilon^s}{s+1}. \end{aligned} \tag{31}$$

It is easy to see that $\omega(f_\varepsilon; \delta) = 1$ for $\delta \geqslant \varepsilon$. Hence, we have

$$\omega(f_\varepsilon; D_N^{1/s}) = 1. \tag{32}$$

From (7), we have

$$R_N(f_\varepsilon) \leqslant c\omega(f_\varepsilon; D_N^{1/s}). \tag{33}$$

Combining (31), (32), and (33) we get

$$1 - \frac{N\varepsilon^s}{s+1} \leqslant c.$$

Passing to the limit as $\varepsilon \to 0+$ in this inequality we obtain (8). This completes the proof of Theorem 2.

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